General Spin Wave Instability Theory
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Abstract—The theory of spin wave instability for magnetic insulators is extended to include generalized anisotropy, ellipsoidal shape, and a microwave pumping field with a general polarization for both first and second order processes. Representative results are given for the instability threshold microwave field amplitude vs. static field and the corresponding critical modes for a Zn–Y hexagonal ferrite thin disk.

Index Terms—Ferrites, magnetic anisotropy, nonlinear magnetics, spin waves.

I. INTRODUCTION

Spin wave instability is a nonlinear process which occurs in magnetic materials when the applied microwave magnetic field reaches some threshold value. At this point, energy is transferred from the microwave field to parametrically excited spin wave pairs. The particular critical modes which are excited at threshold depend on the details of the sample configuration, the static field, the anisotropy, and the pumping configuration. The initial theory of spin wave instability was developed by Suhl [1] for subsidiary absorption and resonance saturation, and by Schlömann et al. [2] for parallel pumping. A cogent review of the full theory is given in [3]. The combination of parallel pumping and subsidiary absorption, termed oblique pumping, was considered in [4], [5]. The effect of anisotropy on instability processes was considered in [4]–[6]. References [4] and [5] presented a generalized tensor approach to anisotropy with specific calculations for the cubic case in the high field limit. In [6] Schlömann et al. considered the case of easy plane hexagonal ferrites.

Most of the previous theories were concerned with one special case or another. None of these works provide a fully general theoretical framework for the analysis of first and second order processes in an insulating material of general shape, with a general anisotropy, or for a possibly noncollinear magnetization and field. The purpose of the work described below was to develop such a theory. The formalism includes provision for a sample of ellipsoidal shape, with a general free energy based magnetic anisotropy, and subject to a general microwave pumping field configuration.

This new formulation is based on the formalism of [1] and the general effective field approach given in [5]. The theory considers only spin waves in the short wavelength limit for which sample surface effects can be neglected. The analysis yields working formulae for the threshold microwave field amplitude for spin wave instability, denoted by $H_{c1}$, as a function of the spin wave decay rate and the spin wave wave vector $k$. The spin wave decay rate is expressed in linewidth units through an equivalent “spin wave linewidth” parameter $\Delta H_k$. The $H_{c1}$ threshold is $k$ dependent. For a specific material, sample shape, static magnetic field, and microwave frequency and pumping configuration, one may determine the minimum $H_{c1}$ relative to all of the available spin wave modes and the corresponding critical mode $k$ for this minimum. This minimum threshold, denoted as $H_{c2}$, should then correspond to the microwave threshold observed experimentally.

Section II describes the theoretical approach, but without detailed formulae. The full theory and a detailed development of all working equations will be published separately. Section III presents an example of the analysis for a particular case of current technological interest, oblique pumping for first order processes in an easy plane magnetic insulator with parameters which match those for Zn–Y hexagonal ferrite.

II. THEORY

Consider an ellipsoidal-shaped single domain ferrite sample in an externally applied uniform static magnetic field $H_{\text{ext}}$ and a microwave pumping field $h$. One starts with the torque equation of motion for the general magnetization vector $M$ [7],

$$\frac{dM}{dt} = -|\gamma| M \times H_{\text{eff}}, \quad (1)$$

where $|\gamma|$ is the absolute value of the electron gyromagnetic ratio and $H_{\text{eff}}$ is the total effective magnetic field. The effective field $H_{\text{eff}}$ consists of the external static field $H_{\text{ext}}$, the microwave pumping field $h$, the demagnetizing field, and effective fields due to anisotropy and exchange. The demagnetizing and anisotropy fields contain both static and dynamic components. These two general fields may be written in the form $N \cdot 4\pi M$ and $A \cdot M$, respectively, where $N$ and $A$ are tensors. The effective exchange field has dynamic components only.

Initially, (1) is developed in a reference frame defined by the principal axes of the ellipsoidal sample. For simplicity, the anisotropy axes may be taken to coincide with these axes as well, with the $A$ tensor developed accordingly. Alternatively, one may use a completely general anisotropy and develop the $A$ accordingly. For any given value of the static field $H_{\text{ext}}$, the static magnetization $M_s$ will not necessarily be lined up with $H_{\text{ext}}$. The static effective field, however, will always be parallel to $M_s$ under conditions of static equilibrium. It is convenient, therefore, to work in a so-called precessional frame in which the $z$-direction coincides with the direction of $M_s$. This direction is determined through the usual static equilibrium considerations. One then transforms the magnetization vector and all fields into...
the precessional frame. This automatically yields specific equations for which all static fields are along \( z \). One has dynamic components of \( \mathbf{M} \) and \( \mathbf{H}_{\text{ext}} \) both along \( z \) and transverse to \( z \).

Based on the precessional frame formulation, one can now obtain the equations of motion for the spin wave amplitudes. One simply does a plane wave Fourier expansion for the dynamic components of \( \mathbf{M} \) in the precessional frame. One then considers specific pairs of terms in this expansion for \( \pm \mathbf{k} \), where \( \mathbf{k} \) is a general spin wave wave vector. Terms with \( k = 0 \) correspond to the uniform mode. Such terms, when present, are connected with components of the microwave field which are transverse to the \( z \)-direction.

Through the above procedure, one obtains coupled nonlinear equations for the amplitudes of particular spin waves with wave vectors \( \mathbf{k} \) and \( -\mathbf{k} \), taken as \( \alpha_{\mathbf{k}} \) and \( \alpha_{-\mathbf{k}} \). These equations have the form

\[
-\mathbf{i} \partial_t \alpha_{\mathbf{k}} = (A_{\mathbf{k}} + C_{\mathbf{k}}) \alpha_{\mathbf{k}} + (B_{\mathbf{k}} + D_{\mathbf{k}}) \alpha_{-\mathbf{k}}^*.
\] (2)

The \( A_{\mathbf{k}} \) and \( B_{\mathbf{k}} \) represent linear terms. If one applies the Holstein–Primakoff transformation [7] to the linear terms in (2) only, one obtains the appropriate spin wave dispersion relation for the spin wave frequency \( \omega_{\mathbf{k}} \) vs. \( \mathbf{k} \). If one applies this same transformation to (2) in its entirety, one obtains a new equation of motion of the form

\[
-\mathbf{i} \partial_t \mathbf{b}_h = \omega_{\mathbf{k}} \mathbf{b}_h + F_{\mathbf{k}} \mathbf{b}_h + G_{\mathbf{k}} b_{-\mathbf{k}}^*.
\] (3)

The \( F_{\mathbf{k}} \) and \( G_{\mathbf{k}} \) coefficients in (3) contain terms which include \( a_0 \) factors and components of the internal microwave field in various combinations.

As discussed in [3], the \( G_{\mathbf{k}} \) term in (3) is responsible for the spin wave instability. The \( G_{\mathbf{k}} \) coefficient increases with the microwave pumping field. Spin wave instability occurs when \( G_{\mathbf{k}} \) exceeds the spin wave relaxation rate \( \eta_{\mathbf{k}} \). The condition \( G_{\mathbf{k}} = \eta_{\mathbf{k}} \) determines the spin wave instability threshold \( h_{\text{cr}} \) for the given spin wave modes at \( \omega_{\mathbf{k}} \) and \( \pm \mathbf{k} \). One then seeks modes out of the ensemble of available spin wave modes with the lowest threshold for the given field, sample parameters, and pumping geometry under consideration. This threshold field is denoted as \( h_{\text{cr}} \). The \( \omega_{\mathbf{k}} \) and \( \mathbf{k} \) values for this minimum threshold define the critical mode which corresponds to this \( h_{\text{cr}} \). The spin wave linewidth parameter introduced above is defined by \( \Delta H_{\mathbf{k}} = 2|\gamma_{\mathbf{k}}/\eta_{\mathbf{k}}| \).

The analysis shows that there are two particular spin wave frequencies which can yield the minimum threshold condition, \( \omega_{\mathbf{k}} = \omega_{\mathbf{p}} \) and \( \omega_{\mathbf{k}} = \omega_{\mathbf{p}}/2 \), where \( \omega_{\mathbf{p}} \) is the pumping frequency. The \( \omega_{\mathbf{k}} = \omega_{\mathbf{p}}/2 \) condition corresponds to so-called first order processes because the microwave field amplitude \( h_{\text{cr}} \) occurs to the first power in the relevant terms in \( G_{\mathbf{k}} \). The \( \omega_{\mathbf{k}} = \omega_{\mathbf{p}} \) condition corresponds to second order processes because \( h_{\text{cr}} \) occurs to the second power in the relevant \( G_{\mathbf{k}} \) terms. Because of the \( \omega_{\mathbf{k}} = \omega_{\mathbf{p}}/2 \) condition, first order processes give rise to nonlinear effects for static fields which are typically well below the field range for ferromagnetic resonance (FMR). For second order processes with \( \omega_{\mathbf{k}} = \omega_{\mathbf{p}} \), the important effects usually occur at fields which are close to the FMR field. If the frequency is low enough, these regions may coincide.

Fig. 1 shows the calculated spin wave instability threshold field \( h_{\text{cr}} \) and the corresponding critical modes as a function of \( H_{\text{ext}} \) for first order processes only. The control parameter was the magnetic field angle \( \theta_H \). The calculations were done for parameters which match realistic experimental situations for \( \text{Zn–Y} \). \( 4\pi M_0 = 2800 \, \text{G}, \omega_p/2\pi = 9 \, \text{GHz}, |\gamma| = 2.8 \, \text{GHz/kOe}, \) an anisotropy field \( H_A = 9.0 \, \text{kOe}, \) and an exchange stiffness field parameter \( D = 5 \times 10^{-6} \text{Oe cm}^2/\text{rad}^2 \).

The anisotropy field \( H_A \) is defined by \( H_A = 2 |K_u|/M_s \), where \( K_u \) is the uniaxial anisotropy energy constant in erg/cm\(^3\). The results shown below are for the spin wave instability threshold field \( h_{\text{cr}} \) and the corresponding critical modes as a function of \( H_{\text{ext}} \) for first order processes only. The control parameter was the magnetic field angle \( \theta_H \). The calculations were done for parameters which match realistic experimental situations for \( \text{Zn–Y} \). \( 4\pi M_0 = 2800 \, \text{G}, \omega_p/2\pi = 9 \, \text{GHz}, |\gamma| = 2.8 \, \text{GHz/kOe}, \) an anisotropy field \( H_A = 9.0 \, \text{kOe}, \) and an exchange stiffness field parameter \( D = 5 \times 10^{-6} \text{Oe cm}^2/\text{rad}^2 \).

Fig. 2 shows the calculated spin wave instability threshold field \( h_{\text{cr}} \) and the corresponding critical modes as a function of \( H_{\text{ext}} \) for first order processes in the easy plane disk at a pumping frequency of 9 GHz. The calculations were done for a full range of the field angle relative to the disk normal, \( \theta_H \), from zero degrees to 90°, relative to the disk normal. Specific results in the figure are for
Fig. 2. Threshold and critical mode parameters as a function of the external field $H_{\text{ext}}$ for the easy plane disk with 9 GHz microwave excitation. The graphs show (a) first order spin wave instability threshold field vs. $H_{\text{ext}}$, (b) critical mode spin wave wave number vs. $H_{\text{ext}}$, and (c) polar spin wave propagation angle vs. $H_{\text{ext}}$. The graphs show results for values of $\theta_H$, the field angle relative to the disk normal, as indicated. The spin wave linewidth $\Delta k_{\text{w}}$ was set at 1 Oe for all computations.

The results in Fig. 2 are self explanatory, but several points of emphasis are important. 1) The figure does not indicate the values of the azimuthal spin wave propagation angle $\varphi_k$ for the critical modes. The calculations yielded $\varphi_k = -90^\circ$ over the entire range of $H_{\text{ext}}$ values considered here. This critical mode $\varphi_k$ value corresponds to the highest ellipticity for the mode and, hence, the lowest threshold. 2) When $H_{\text{ext}}$ is very small, all curves in Fig. 2 start from the same values. This limit corresponds to the same physical situation with $\mathbf{M}_S$ along $Y$. 3) The curves for $\theta_H = 90^\circ$ correspond to parallel pumping and the results are in complete agreement with [6]. 4) There is a clear drop in $\theta_h$ and increase in $h_{\text{crit}}$ as $H_{\text{ext}}$ is increased up to the points where the critical mode $k$-values become zero. This is due to the gradual change from parallel to oblique pumping as $\mathbf{M}_S$ is pulled away from $Y$.

The influence of the frequency and the direction of the pumping field on $h_{\text{crit}}$ was also investigated. An increase in $\omega_p$ leads to the increase of $h_{\text{crit}}$, with graphs which are similar to those in Fig. 2. A change in the direction of $\mathbf{h}$ results in significant changes in $h_{\text{crit}}$ and the critical modes. When $\mathbf{h}$ is parallel to $Z$ instead of $Y$, $h_{\text{crit}}$ is about two times greater than in Fig. 2. When $\mathbf{h}$ is parallel to $X$, $h_{\text{crit}}$ is slightly lower than in Fig. 2. This result is surprising. In isotropic ferrites, parallel pumping always yields the lowest $h_{\text{crit}}$.

REFERENCES